Modern approaches to quantum gravity

Homework 7 — Fall 2025

1. Spinning fields

Consider a general symmetry transformation acting both on the spacetime manifold and on fields as

$$x \to x'(x) \equiv f(x), \qquad \mathcal{O}(x) \to \mathcal{O}'(x') \equiv F[\mathcal{O}(x)] \tag{1}$$

Then we have the Ward Identity in its finite form (see section 4.2.3 of Di Francesco)

$$\langle \mathcal{O}_1'(x_1)...\mathcal{O}_n'(x_n)\rangle = \langle \mathcal{O}_1(x_1)...\mathcal{O}_n(x_n)\rangle \tag{2}$$

(a) Using the conformal Ward identities for translations, boosts and dilatations, show that the two-point function of a vector primary operator j_μ in a CFT is given by

$$\langle j_\mu(x) j_\nu(y)\rangle = \frac{C_j}{|x-y|^{2\Delta}} I_{\mu\nu}(x-y) \tag{3}$$

with

$$I_{\mu\nu}(x) = \eta_{\mu\nu} + B\frac{x_\mu x_\nu}{x^2} \tag{4}$$

where B is a constant that we will determine later.

A tensor primary field of scaling dimension Δ and spin J transforms under conformal transformation as follows

$$T'_{\mu_1...\mu_J}(x) = \left|\frac{\partial x'}{\partial x}\right|^{\frac{\Delta-J}{d}} \frac{\partial x'^{\nu_1}}{\partial x^{\mu_1}} ... \frac{\partial x'^{\nu_J}}{\partial x^{\mu_J}} T_{\nu_1...\nu_J}(x') \tag{5}$$

(b) Using infinitesimal conformal transformations $x'^\mu = x^\mu + \xi^\mu$, now derive all the conformal Ward identities for the vector two-point function as differential equations. In particular, show that the Ward identity associated to special conformal transformations $\xi^\mu = 2(b\cdot x)x^\mu - x^2 b^\mu$ for $\langle j_\mu(x) j_\nu(0)\rangle$ (we set $y=0$) determines $B=-2$.

(c) *(Optional)* Alternatively, you can show that the Ward identity for discrete inversion transformations $x'^\mu = \frac{x^\mu}{x^2}$ imposes $B=-2$.
Note that special conformal transformations can be expressed as a combination of inversions and translations as $SCT = Inv. \circ Transl. \circ Inv.$
Hint 1: show first that for such choice of B, we have

$$I_\mu^\rho(x) I_\rho^\nu(x) = \delta_\mu^\nu$$
$$\frac{\partial x'^\nu}{\partial x'^\mu} = \frac{I_\mu^\nu(x)}{x^2}$$
$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{x^{2d}}$$

Hint 2: in the final steps you might find it useful to employ translation invariance to set $y \to 0$, hence $y' \to \infty$ and $I_{\mu\nu}(x'-y') \to I_{\mu\nu}(y')$ in the Ward identity.

(d) Show that if j_μ is a conserved current, then its scaling dimension must be $\Delta = d-1$. Interpret this result physically.

(e) The two-point function of a spin 2 (symmetric and traceless) tensor primary operator in a CFT can be similarly determined to take the form

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(y)\rangle = \frac{C_T}{|x-y|^{2\Delta}}\, I_{\mu\nu,\rho\sigma}(x-y)$$

where $I_{\mu\nu,\rho\sigma}(x-y)$ is constructed from $I_{\mu\nu}(x-y)$ as:

$$I_{\mu\nu,\rho\sigma}(x-y) = \frac{1}{2}\left(I_{\mu\rho}(x-y)I_{\nu\sigma}(x-y) + I_{\mu\sigma}(x-y)I_{\nu\rho}(x-y)\right) - \frac{1}{d}\eta_{\mu\nu}\eta_{\rho\sigma}$$

Show that a conserved stress tensor must have $\Delta = d$, and interpret the result.

2. Introduction to the embedding space formalism

The Euclidean conformal group of a d-dimensional CFT is $SO(d+1,1)$. This makes it natural to consider a formalism where conformal symmetry is realized as $d+2$ dimensional Lorentz symmetry. The idea is the following. Consider a $d+1$-dimensional light-cone embedded in $d+2$ dimensions with coordinates $P^0, ..., P^{d+1}$, namely

$$ds^2 = \eta^{AB} dP_A dP_B \tag{6}$$

$$-(P^0)^2 + (P^1)^2 + ... + (P^{d+1})^2 = 0 \tag{7}$$

where $A, B = 0, ..., d+1$. A transverse section of this light-cone is a d-dimensional manifold where the CFT lives. For example, one can consider the section called the *Poincaré* section $P^0 + P^{d+1} = 1$ of the light-cone. This section can be parameterized by $x^\mu \in \mathbb{R}^d$, where $\mu = 1, ..., d$, namely

$$P^0(x) = \frac{1+x^2}{2} \quad P^\mu(x) = x^\mu, \quad P^{d+1}(x) = \frac{1-x^2}{2} \tag{8}$$

(a) Show that the induced metric is flat on this section of the light-cone.

(b) Show that a rescaled parametrization $P \to \Omega(x)P$ corresponds to an induced metric which is Weyl rescaled $ds^2 = \Omega^2(x)\delta_{\mu\nu}dx^\mu dx^\nu$. This makes it natural to extend a primary operator $\mathcal{O}$ from the Poincaré section to the full light-cone with the **homogeneity** property

$$\mathcal{O}(\lambda P) = \lambda^{-\Delta}\mathcal{O}(P) \tag{9}$$

(c) Impose $SO(d+1,1)$ Lorentz invariance and homogeneity on the two point function $\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\rangle$ and on the three point function $\langle \mathcal{O}(P_1)\mathcal{O}(P_2)\mathcal{O}(P_3)\rangle$ where $\mathcal{O}$ is a primary operator. Show that you recover the form expected by conformal symmetry when writing them in the Poincaré section.

(d) Vector primary operators are also extended to embedding space by imposing

$$P^A \mathcal{O}_A(P) = 0 \quad \mathcal{O}_A(\lambda P) = \lambda^{-\Delta}\mathcal{O}_A(P) \tag{10}$$

Note that this implies a redundancy

$$\mathcal{O}_A \sim \mathcal{O}_A + P_A \Lambda(P) \tag{11}$$

for any scalar operator such that $\Lambda(\lambda P) = \lambda^{-\Delta-1}\Lambda(P)$. The physical operator is obtained by projecting the $d+2$ component-vector to the section

$$\mathcal{O}_\mu(x) = \frac{\partial P^A}{\partial x^\mu}\mathcal{O}_A(P)\Big|_{P^A=P^A(x)} \tag{12}$$

Show that the two-point function of vector primary operators is given by

$$\langle \mathcal{O}^A(P_1)\mathcal{O}^B(P_2)\rangle = \text{const}\frac{\eta^{AB}(P_1 \cdot P_2) - P_2^A P_1^B}{(-2P_1 \cdot P_2)^{\Delta+1}} \tag{13}$$

up to redundant terms.

(e) Project this relation to physical operators in order to re-derive the result of the first part of the homework.